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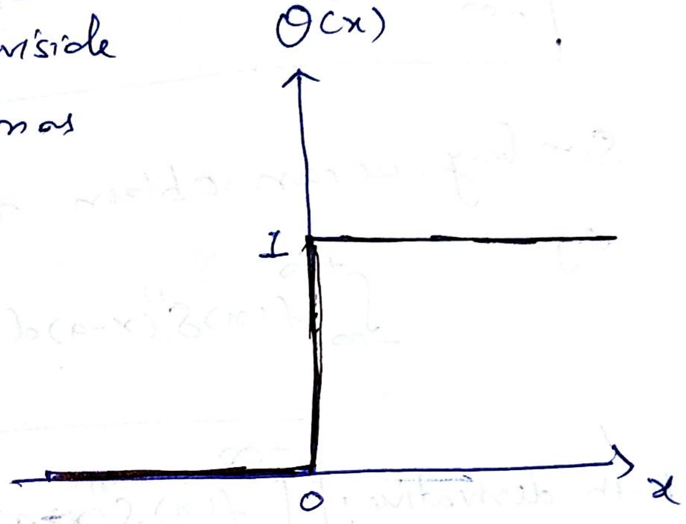
(Classical Electrostatics -03)

Delta function continued--

Derivative of the Delta function:

First, we define the ~~Hea~~ Heaviside function or the step function as



$$\theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0 \end{cases}$$

The derivative of Step function $\theta(x)$ gives the delta function

$$\frac{d\,\theta(x)}{dx} = \delta(x)$$

For derivative of delta function we use Fourier transform and write $\frac{d}{dx}(\delta(x))$ as.

$$\frac{d\delta(x)}{dx} = \delta'(x) = \frac{1}{2\pi}\frac{d}{dx}\int_{-\infty}^{+\infty} e^{ikx}\,dk.$$

Or $$\boxed{\delta'(x) = \frac{d\delta(x)}{dx} = \frac{i}{2\pi}\int_{-\infty}^{+\infty} k\,e^{ikx}\,dk.}$$

Next, we can also obtain derivative in another way by doing integration by parts of $\delta'(x-a)$.

$$\int_{-\infty}^{+\infty} f(x)\,\delta'(x-a)\,dx = f(x)\,\delta(x-a)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f'(x)\,\delta(x-a)\,dx$$

$$= 0 - f'(a)$$

$$\boxed{\int_{-\infty}^{+\infty} f(x)\,\delta'(x-a)\,dx = -f'(a)}$$

Similarly, we can obtain n-th derivation of δ-function e.g.

$$\int_{-\infty}^{+\infty} f(x)\,\delta''(x-a)\,dx = (-1)^2 f''(a) = f''(a)$$

n-th derivative: $\boxed{\int_{-\infty}^{+\infty} f(x)\,\delta^{(n)}(x-a)\,dx = (-1)^n f^{(n)}(a)}$

Here $\delta^{(n)}(x-a) = \dfrac{d^n(\delta(x-a))}{dx^n}$

$$f^{(n)}(a) = \frac{d^n[f(x)]}{dx^n}\Bigg|_{x=a}$$

From n-th derivative result, if we put $f(x)=1$, $n=1$, we obtain,

$$\int_{-\infty}^{+\infty} \delta'(x-a)\,dx = 0$$

Some more properties of derivative of delta function.

(i) $\delta'(x) = -\delta'(-x)$

(ii) $x\,\delta'(x) = -\delta(x)$

(iii) $x^2\delta'(x) = 0$

(iv) $x^2\delta''(x) = 2\delta(x)$. etc.

Similarly one can obtain expressions for higher derivatives

Three-dimensional delta function:

$$\boxed{\delta^3(\vec{r}) = \delta(x)\,\delta(y)\,\delta(z),}$$

where $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$$\int \delta^3(\vec{r})\,d\tau = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz\; \delta(x)\delta(y)\delta(z) = 1$$

And $\int \delta^3(\vec{r}-\vec{a})\, f(\vec{r})\, d\tau = f(\vec{a})$

The other way of writing it.

$$\boxed{\delta(\vec{r}-\vec{r}\,') = \delta(x-x')\,\delta(y-y')\,\delta(z-z')}$$

In spherical coordinates, δ-function is given by

$$\delta(\vec{r}-\vec{r}\,') = \frac{1}{r^2}\delta(r-r')\,\delta(\cos\theta-\cos\theta')\,\delta(\phi-\phi')$$

$$= \frac{1}{r^2\sin\theta}\,\delta(r-r')\,\delta(\theta-\theta')\,\delta(\phi-\phi')$$

Here we have used property of δ-function

$$\delta(\cos\theta-\cos\theta') = \frac{\delta(\theta-\theta')}{\sin\theta}$$

Fourier transform of 3d delta function.

$$\delta(\vec{r}-\vec{r}') = \frac{1}{(2\pi)^3}\int d^3k\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

Two important relations encountered in electrodynamics involving δ-function –

$$\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta(\vec{r})$$

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(\vec{r})$$

Dimensions: Physical dimension $= \dfrac{1}{[\text{dimension of its argument}]}$

For 1-d, δ-function $[\delta(x)] = \dfrac{1}{[x]} = \dfrac{1}{[L]}$

2-d – $[\delta(x)\,\delta(y)] = \dfrac{1}{[L][L]} = \dfrac{1}{[L^2]}$

3-d – $[\delta(\vec{r})] = [\delta(x)\,\delta(y)\,\delta(z)] = \dfrac{1}{[x]}\cdot\dfrac{1}{[y]}\cdot\dfrac{1}{[z]} = \dfrac{1}{[L^3]}$

H.W: Evaluate the following integrals.

(I) $\int_0^5 \cos x\,\delta(x-\pi)\,dx$

(II) $\int_0^3 x^3\,\delta(x+1)\,dx$

(III) $\int_{-\infty}^{+\infty} \ln(x+3)\,\delta(x+2)\,dx$

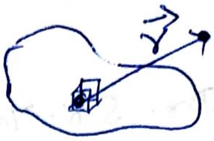
(IV) $\int_{-2}^{2} (2x+3)\,\delta(3x)\,dx$

V $\int_0^2 (x^3+3x+2)\,\delta(1-x)\,dx$

(VI) $\int_{-\infty}^{a} \delta(x-b)\,dx$

Continuous Charge distribution:

Continuous distribution –



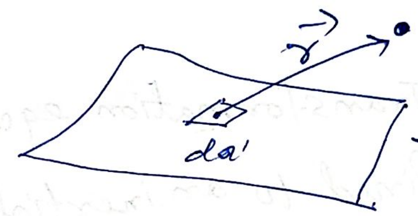
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\hat{r}}{r^2} dq$$

Line charge (λ) –

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_P \frac{\lambda(\vec{r}')}{r^2} \hat{r}\, dl'$$

, $dq \to \lambda dl'$

Surface charge (σ) –



$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_S \frac{\sigma(\vec{r}')}{r^2} \hat{r}\, da'$$

$\to dq \to \sigma da'$

Volume charge (ρ) –

$\to dq \to \rho d\tau'$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r}')}{r^2} \hat{r}\, d\tau'$$

One can also write charge distribution say ($\tilde{\rho}(\vec{x})$) in terms of δ-function.

General form of $\vec{E}$ in terms of $\tilde{\rho}(x)$ –

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} \tilde{\rho}(\vec{x}')\, d\vec{x}'$$

For Point charge :- $\tilde{\rho}(\vec{x}) = q_i \delta^3(\vec{x} - \vec{x}_i)$, $x_i \to$ Position of charge

$\left\{ \vec{r} = \vec{x} - \vec{x}', \quad r = |\vec{x} - \vec{x}'| \right.$

Line charge : $\tilde{\rho}(\vec{x}) = \lambda(\vec{x})\, \delta^{(2)}$

$\delta^2 \to$ 2D delta function

Surface charge : $\tilde{\rho}(\vec{x}) = \sigma(\vec{x})\, \delta^1$

$$\delta^2(\vec{x}) = \delta(x - x_0)\, \delta(y - y_0) \text{ etc}$$